The Gervais-Neveu-Felder equation for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra

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Abstract

Using a contraction procedure, we construct a twist operator that satisfies a shifted cocycle condition, and leads to the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra. The corresponding universal $\mathcal{R}_h(y)$ matrix obeys a Gervais-Neveu-Felder equation associated with the $U_{h;y}(sl(2))$ algebra. For a class of representations, the dynamical Yang-Baxter equation may be expressed as a compatibility condition for the algebra of the Lax operators.

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Recently a class of invertible maps between the classical $sl(2)$ and the non-standard Jordanian $U_h(sl(2))$ algebras has been obtained [1]-[3]. The classical and the Jordanian coalgebraic structures may be related [2]-[5] by the twist operators corresponding to these maps. Following the first twist leading from the classical to the Jordanian Hopf structure, it is possible to envisage a second twist leading to a quasi-Hopf quantization of the Jordanian $U_h(sl(2))$ algebra. By explicitly constructing the appropriate universal twist operator that satisfies a shifted cocycle condition, we here obtain the Gervais-Neveu-Felder (GNF) equation satisfied by the universal $\mathcal{R}$ matrix of a one-parametric quasi-Hopf deformation of the $U_h(sl(2))$ algebra.

The GNF equation corresponding to the standard Drinfeld-Jimbo deformed $U_q(sl(2))$ algebra was studied in the context of Liouville field theory [6], quantization of Kniznik-Zamolodchikov-Bernard equation [7] and the quantization of the Calogero-Moser model in the R matrix formalism [8]. The general construction of the twist operators leading to the GNF equation corresponding to the quasi-triangular standard Drinfeld-Jimbo deformed $U_q(\mathsf{g})$ algebras and superalgebras were obtained in [9]-[11].

For the sake of completeness, we start by enlisting the general properties of a quasi-Hopf algebra $\mathcal{A}$ [12]. For all $a \in \mathcal{A}$ there exist an invertible element $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ and the elements $(\alpha, \beta) \in \mathcal{A}$, such that

$$
\begin{aligned}
(\mathrm{id} \otimes \triangle) \triangle (a) &= \Phi \; (\triangle \otimes \mathrm{id})(\triangle(a)) \; \Phi^{-1}, \\
(\mathrm{id} \otimes \mathrm{id} \otimes \triangle)(\Phi) \; (\triangle \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) &= (1 \otimes \Phi) \; (\mathrm{id} \otimes \triangle \otimes \mathrm{id})(\Phi) \; (\Phi \otimes 1), \\
(\varepsilon \otimes \mathrm{id}) \circ \triangle &= \mathrm{id}, \\
(\mathrm{id} \otimes \varepsilon) \circ \triangle &= \mathrm{id}, \\
\sum_r S(a_r^{(1)}) \; \alpha \; a_r^{(2)} &= \varepsilon(a) \; \alpha, \\
\sum_r a_r^{(1)} \; \beta \; S(a_r^{(2)}) &= \varepsilon(a) \; \beta, \\
\sum_r X_r^{(1)} \; \beta \; S(X_r^{(2)}) \; \alpha \; X_r^{(3)} &= 1, \\
\sum_r S(\bar{X}_r^{(1)}) \; \alpha \; \bar{X}_r^{(2)} \; \beta \; S(\bar{X}_r^{(3)}) &= 1,
\end{aligned}
\tag{1}
$$

where

$$
\triangle (a) = \sum_r a_r^{(1)} \otimes a_r^{(2)}, \quad \Phi = \sum_r X_r^{(1)} \otimes X_r^{(2)} \otimes X_r^{(3)}, \quad \Phi^{-1} = \sum_r \bar{X}_r^{(1)} \otimes \bar{X}_r^{(2)} \otimes \bar{X}_r^{(3)}. \tag{2}
$$

A quasi-triangular quasi-Hopf algebra is equipped with a universal $\mathcal{R}$ matrix satisfying

$$
\begin{aligned}
\triangle^{op} (a) &= \mathcal{R} \; \triangle (a) \; \mathcal{R}^{-1}, \\
(\mathrm{id} \otimes \triangle)(\mathcal{R}) &= \Phi_{231}^{-1} \; \mathcal{R}_{13} \; \Phi_{213} \; \mathcal{R}_{12} \; \Phi_{123}^{-1}, \\
(\triangle \otimes \mathrm{id})(\mathcal{R}) &= \Phi_{312} \; \mathcal{R}_{13} \; \Phi_{132}^{-1} \; \mathcal{R}_{23} \; \Phi_{123}.
\end{aligned}
\tag{3}
$$

The algebra is known as triangular if the additional relation

$$\mathcal{R}_{21} = \mathcal{R}^{-1} \tag{4}$$

is satisfied. In a quasi-triangular quasi-Hopf algebra, the universal $\mathcal{R}$ matrix satisfies quasi-Yang-Baxter equation

$$\mathcal{R}_{12}\ \Phi_{312}\ \mathcal{R}_{13}\ \Phi^{-1}_{132}\ \mathcal{R}_{23}\ \Phi_{123} = \Phi_{321}\ \mathcal{R}_{23}\ \Phi^{-1}_{231}\ \mathcal{R}_{13}\ \Phi_{213}\ \mathcal{R}_{12}. \tag{5}$$

An invertible twist operator $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfying the relation

$$(\varepsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{F}) \tag{6}$$

performs a gauge transformation as follows:

$$\begin{aligned}
\triangle_{\mathcal{F}}(a) &= \mathcal{F}\ \triangle(a)\ \mathcal{F}^{-1},\\
\Phi_{\mathcal{F}} &= \mathcal{F}_{23}(\mathrm{id} \otimes \triangle)(\mathcal{F})\ \Phi\ (\triangle \otimes \mathrm{id})(\mathcal{F}^{-1})\mathcal{F}^{-1}_{12},\\
\alpha_{\mathcal{F}} &= \sum_r S(\bar{f}^{(1)}_r)\ \alpha\ \bar{f}^{(2)}_r,\\
\beta_{\mathcal{F}} &= \sum_r f^{(1)}_r\ \beta\ S(f^{(2)}_r),\\
\mathcal{R}_{\mathcal{F}} &= \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1},
\end{aligned} \tag{7}$$

where

$$\mathcal{F} = \sum_r f^{(1)}_r \otimes f^{(2)}_r, \qquad \mathcal{F}^{-1} = \sum_r \bar{f}^{(1)}_r \otimes \bar{f}^{(2)}_r. \tag{8}$$

The Jordanian Hopf algebra $U_h(sl(2))$ is generated by the elements $(T^{\pm 1}\,(= e^{\pm hX}), Y, H)$, satisfying the algebraic relations [13]

$$[H, T^{\pm 1}] = T^{\pm 2} - 1,\ [H, Y] = -\frac{1}{2}\left(Y(T + T^{-1}) + (T + T^{-1})Y\right),\ [X, Y] = H, \tag{9}$$

whereas the coalgebraic properties are given by [13]

$$\begin{aligned}
\triangle(T^{\pm 1}) &= T^{\pm 1} \otimes T^{\pm 1},\ \triangle(Y) = Y \otimes T + T^{-1} \otimes Y,\ \triangle(H) = H \otimes T + T^{-1} \otimes H,\\
\varepsilon(T^{\pm 1}) &= 1, \qquad \varepsilon(Y) = \varepsilon(H) = 0,\\
S(T^{\pm 1}) &= T^{\mp 1}, \quad S(Y) = -TYT^{-1}, \quad S(H) = -THT^{-1}.
\end{aligned} \tag{10}$$

The universal $\mathcal{R}_h$ matrix of the triangular Hopf algebra $U_h(sl(2))$ is given in a convenient form [14] by

$$\mathcal{R}_h = \exp(-hX \otimes TH)\ \exp(hTH \otimes X). \tag{11}$$

An invertible nonlinear map of the generating elements of the $U_h(sl(2))$ algebra on the elements of the classical $U(sl(2))$ algebra plays a pivotal role in the present work. The map reads [2]

$$T = \tilde{T}, \quad Y = J_- - \frac{1}{4} h^2\, J_+\,(J_0^2 - 1), \quad H = \left(1 + (hJ_+)^2\right)^{1/2} J_0, \tag{12}$$

where $\tilde{T} = hJ_+ + \left(1 + (hJ_+)^2\right)^{1/2}$. The elements $(J_\pm, J_0)$ are the generators of the classical $sl(2)$ algebra

$$[J_0, J_\pm] = \pm 2\, J_\pm, \qquad [J_+, J_-] = J_0. \tag{13}$$

The twist operator specific to the map (12), transforming the trivial classical $U(sl(2))$ coproduct structure to the non-cocommuting coproduct properties (10) of the Jordanian $U_h(sl(2))$ algebra, has been obtained [3], [4] as a series expansion in powers of h. The transforming operator between the two above-mentioned antipode maps has been obtained [4] in a closed form.

Our present derivation of the GNF equation corresponding to the Jordanian $U_h(sl(2))$ algebra closely parallels the description in [8]. These authors obtained the solutions of the GNF equation in the case of the standard Drinfeld-Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra by constructing the universal twist operator depending on a parameter x :

$$\begin{aligned}\mathcal{F}(x) \;&=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{(q-q^{-1})^k}{[k]_q\,!}\;x^{2k}\,q^{k(k+1)/2}\Big[\prod_{l=1}^{k}\left(1\otimes 1\;-\;x^2q^{2l}\;1\otimes q^{2\mathcal{J}_0}\right)^{-1}\Big]\;\times\\ &\times\;\; q^{\frac{k}{2}\,\mathcal{J}_0}\;\mathcal{J}_+^k\;\otimes\;q^{\frac{3k}{2}\,\mathcal{J}_0}\;\mathcal{J}_-^k\,,\end{aligned} \tag{14}$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. The generators of the $U_q(sl(2))$ algebra satisfies [12] the relations

$$q^{\mathcal{J}_0}\;\mathcal{J}_\pm\;q^{-\mathcal{J}_0} = q^{\pm 2}\;\mathcal{J}_\pm, \quad [\mathcal{J}_+, \mathcal{J}_-] = [\mathcal{J}_0]_q. \tag{15}$$

A key ingredient in our method is the contraction technique developed in [2], where a matrix G

$$G = E_q(\eta\mathcal{J}_+)\;\otimes\;E_q(\eta\mathcal{J}_+), \quad \eta = \frac{h}{q-1} \tag{16}$$

performs a similarity transformation on the universal $\mathcal{R}_q$ matrix of the $U_q(sl(2))$ algebra [12]. The twisted exponential $E_q(\chi)$ reads

$$E_q(\chi) = \sum_{n=0}^{\infty}\frac{\chi^n}{[n]_q\,!}. \tag{17}$$

The transforming matrix G is singular in the $q \to 1$ limit. The transformed $R_h^{j_1;j_2}$ matrix for an arbitrary $(j_1; j_2)$ represention

$$R_h^{j_1;j_2} = \lim_{q\to 1}\left[G^{-1}\,R_q^{j_1;j_2}\,G\right] \tag{18}$$

is, however, nonsingular and coincide, on account of the map (12), with the result obtained directly from the expression (11) of the universal $\mathcal{R}_h$ matrix. In the above contraction process the following two identities play a crucial role:

$$\begin{aligned}(E(\eta\,\mathcal{J}_+))^{-1}\,q^{\alpha\,\mathcal{J}_0\,/\,2}\,E(\eta\,\mathcal{J}_+) \;&=\; \mathcal{T}_{(\alpha)}\,q^{\alpha\,\mathcal{J}_0\,/\,2},\\ (E(\eta\,\mathcal{J}_+))^{-1}\,\mathcal{J}_-\,E(\eta\,\mathcal{J}_+) \;&=\; -\,\frac{\eta}{q-q^{-1}}\,(\,\mathcal{T}_{(1)}\,q^{\mathcal{J}_0}\;-\;\mathcal{T}_{(-1)}\,q^{-\mathcal{J}_0})\;+\;\mathcal{J}_-,\end{aligned} \tag{19}$$

where $\mathcal{T}_{(\alpha)} = (E(\eta\,\mathcal{J}_+))^{-1}\,E(q^\alpha\eta\,\mathcal{J}_+)$. In the $q \to 1$ limit, it may be proved [2]

$$\lim_{q\to 1}\,\mathcal{T}_{(\alpha)} = \tilde{T}^\alpha = T^\alpha. \tag{20}$$

The second equality in (20) follows from the map (12).

Using the contraction scheme discussed above we now obtain an one-parametric twist operator $\mathcal{F}_h(y) \in U_h(sl(2)) \otimes U_h(sl(2))$, which satisfies a shifted cocycle condition. The twist operator $\mathcal{F}_h(y)$ gauge transforms *à la* (7) the Jordanian Hopf algebra

$U_h(sl(2))$ to a quasi-Hopf $U_{h;y}(sl(2))$ algebra and the transformed universal $\mathcal{R}_h(y)$ matrix satisfies the corresponding GNF equation. To this end we first compute

$$\tilde{\mathcal{F}}(y) = \lim_{q\to 1}\,(G^{-1}\,\mathcal{F}(x)G)_{x^2=y\,(q-1)}\,, \tag{21}$$

where $\mathcal{F}(x)$ is given by (14).

A new feature here is the reparametrization described by

$$y = \frac{x^2}{q-1}, \tag{22}$$

which is necessary for obtaining *nonsingular* result in the $q \to 1$ limit. In (22) we assume that $x \to 0$ in the $q \to 1$ in such a way that y remains finite. Following the above procedure in the said limit we obtain

$$\tilde{\mathcal{F}}(y) = \sum_{k=0}^{\infty} \frac{(hy)^k}{k!}\,\left(\tilde{T}J_+\right)^k \otimes \left(\tilde{T}^3(\tilde{T} - \tilde{T}^{-1})\right)^k. \tag{23}$$

The rhs of (23) is interpreted on account of the map (12) as an element of $U_h(sl(2)) \otimes U_h(sl(2))$. Identifying this in the above sense with the twist operator $\mathcal{F}_h(y)\left(= \tilde{\mathcal{F}}(y)\right)$ we now obtain the crucial result

$$\mathcal{F}_h(y) = \exp\left(\frac{y}{2}\,(1 - T^2)\,\otimes\,(T^2 - T^4)\right). \tag{24}$$

The above twist operator $\mathcal{F}_h(y)$ satisfies the property (6). Following the arguments in [8] we express $\mathcal{F}_h(y)$ as a shifted coboundary

$$\mathcal{F}_h(y) = \triangle(\mathcal{M}(y))\,\left(1 \otimes \mathcal{M}^{-1}(y)\right)\,\left(\mathcal{M}^{-1}(y\,T^4_{(2)}) \otimes 1\right), \tag{25}$$

where the expression for the boundary reads

$$\mathcal{M}(y) = \exp\left(\frac{y}{2}(1 - T^2)\right). \tag{26}$$

The operator $\mathcal{F}_h(y)$ given by (24) satisfies the following shifted cocycle condition

$$(1 \otimes \mathcal{F}_h(y))\,[(\mathrm{id} \otimes \triangle)\,\mathcal{F}_h(y)] = \left(\mathcal{F}_h(y\,T^4_{(3)}) \otimes 1\right)\,[(\triangle \otimes \mathrm{id})\,\mathcal{F}_h(y)]\,. \tag{27}$$

Following (7) the transformed coproduct property may now be read as

$$\triangle_y\,(a) = \mathcal{F}_h(y)\ \triangle\,(a)\ \mathcal{F}_h^{-1}(y) \quad \text{for all } a \in U_{h;y}(sl(2)). \tag{28}$$

It may now be shown that the shifted cocycle condition is a consequence of the following shifted coassociativity property:

$$(\text{id} \otimes \triangle_y) \circ \triangle_y(a) = \left(\triangle_{y\,T^4_{(3)}} \otimes \text{id}\right) \circ \triangle_y(a). \tag{29}$$

Following (7) the gauge-transformed universal $\mathcal{R}_h(y)$ matrix for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra reads

$$\mathcal{R}_h(y) = \mathcal{F}_{h\ 21}(y)\,\mathcal{R}_h\,\mathcal{F}_h^{-1}(y). \tag{30}$$

The coassociator $\Phi(y)$ corresponding to the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra may be obtained for the above constuction of the twist operator obeying the shifted cocycle condition (27). Using (7), (24) and (27) we obtain

$$\begin{aligned} \Phi(y) &= \mathcal{F}_{h\ 12}(y\,T^4_{(3)})\ \mathcal{F}^{-1}_{h\ 12}(y) \\ &= \exp\left[-\frac{y}{2}(1-T^2) \otimes (T^2 - T^4) \otimes (1-T^4)\right]. \end{aligned} \tag{31}$$

The elements $\alpha(y)$ and $\beta(y)$, characterizing the antipode map of the $U_{h;y}(sl(2))$ algebra may be similarly obtained from (7), (10) and (24):

$$\alpha(y) = \exp\left[\frac{y}{2}\,(1-T^2)^2\right], \qquad \beta(y) = \exp\left[-\frac{y}{2}\,(1-T^{-2})^2\right]. \tag{32}$$

Using the guage transformation property of the universal $\mathcal{R}$ matrix in (7) and our construction (24) of the twist operator, we now discuss the GNF equation associated with the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra. The relations (7), (24) and (31) lead to the transformation property

$$\mathcal{R}_{h\ 12}\left(y\,T^4_{(3)}\right) = \Phi_{213}(y)\,\mathcal{R}_{h\ 12}\,(y)\,\Phi^{-1}_{123}(y). \tag{33}$$

Now the quasitriangularity property of $U_{h;y}(sl(2))$ algebra implies via (3), (31) and (33) the following relations:

$$\begin{aligned} &(\text{id} \otimes \triangle_y)\,\mathcal{R}_h\,(y) = \mathcal{F}_{h\ 23}(y)\,\mathcal{F}^{-1}_{h\ 23}\left(y\,T^4_{(1)}\right)\,\mathcal{R}_{h\ 13}(y)\,\mathcal{R}_{h\ 12}\left(y\,T^4_{(3)}\right), \\ &(\triangle_y \otimes \text{id})\,\mathcal{R}_h\,(y) = \mathcal{R}_{h\ 13}\left(y\,T^4_{(2)}\right)\,\mathcal{R}_{h\ 23}(y)\,\mathcal{F}_{h\ 12}\left(y\,T^4_{(3)}\right)\,\mathcal{F}^{-1}_{h\ 12}(y). \end{aligned} \tag{34}$$

Using the transformation property (33) we may now recast the quasi Yang-Baxter equation (5) as the GNF equation associated with the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra:

$$\mathcal{R}_{h\ 12}(y)\,\mathcal{R}_{h\ 13}\left(y\,T^4_{(2)}\right)\,\mathcal{R}_{h\ 23}(y) = \mathcal{R}_{h\ 23}\left(y\,T^4_{(1)}\right)\,\mathcal{R}_{h\ 13}(y)\mathcal{R}_{h\ 12}\left(y\,T^4_{(3)}\right). \tag{35}$$

We now briefly consider the solutions of the above GNF equation (35). Using the universal $\mathcal{R}_h(y)$ matrix (30), the twist operator $\mathcal{F}_h(y)$ in (24) and the map (12) of the generators of the $U_h(sl(2))$ algebra on the corresponding classical elements, we may construct solutions of the GNF equation (35). As illusrations we describe the represention $R_h(y)$ for the $\frac{1}{2} \otimes j$ and the $1 \otimes j$ cases. A $(2j+1)$ dimensional representation of the classical $sl(2)$ algebra (13)

$$\begin{aligned} J_+|jm\rangle &= (j-m)(j+m+1)\,|j\,m+1\rangle, \qquad J_-|jm\rangle = |j\,m-1\rangle, \\ J_0|jm\rangle &= m\,|jm\rangle, \end{aligned} \tag{36}$$

now, via the map (12), immediately furnishes the corresponding $(2j+1)$ dimensional representation of the $U_h(sl(2))$ algebra (9). For the $j=\frac{1}{2}$ case, the generators remain undeformed. For the $j=1$ case, we list the representation of $U_h(sl(2))$ below.

$$\begin{aligned} &(j=1) \\ &X = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & \frac{1}{2}h^2 & 0 \\ 1 & 0 & -\frac{3}{2}h^2 \\ 0 & 1 & 0 \end{pmatrix}, \\ &H = \begin{pmatrix} 2 & 0 & -4h^2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{37}$$

Using the above representations in the expression (30) of the universal $\mathcal{R}_h(y)$ matrix,we obtain

$$R_h^{\frac{1}{2};j}(y) = \begin{pmatrix} T & -hH + \frac{1}{2}h\,(T-T^{-1})\big(1+2y(1-T^4)\big) \\ 0 & T^{-1} \end{pmatrix} \tag{38}$$

and

$$R_h^{1;j}(y) = \begin{pmatrix} T^2 & A & B \\ 0 & 1 & C \\ 0 & 0 & T^{-2} \end{pmatrix}, \tag{39}$$

where

$$\begin{aligned} A &= -2h\,TH - 2hy\,(1-T^2)\,(1-T^4), \\ B &= -2h^2\left[T^2 - T^{-2} - 2\,TH(1-T^{-2}) - (TH)^2 T^{-2}\right] - 4h^2 y\,(1-T^2)\,(1+4T^{-2}-T^4) \\ &\quad -4h^2 y\,TH\,(1-T^2)\,(T^2-T^{-2}) + 2h^2y^2\,(T-T^{-1})^2\,(1-T^4)^2, \\ C &= -2h\,(1-T^{-2}+THT^{-2}) + 2hy\,(1-T^2)(T^2-T^{-2}). \end{aligned} \tag{40}$$

From (38) it follows that the $R_h^{\frac{1}{2};\frac{1}{2}}$ matrix for the fundamental $(1/2;1/2)$ case does not depend on the parameter y. The $R_h(y)$ matrices for the higher representations, however, nontrivially depend on y. The $R_h(y)$ matrices satisfy an "exchange symmetry" between the two sectors of the tensor product spaces:

$$\Big(R_h^{j_1;\,j_2}(y)\Big)_{km,\,ln} = \Big(R_{-h}^{j_2;\,j_1}(y)\Big)_{mk,\,nl}. \tag{41}$$

In the remaining part of the present work we recast the Jordanian GNF equation (35) as a compatibility condition for the algebra of L operators. Using a new parametrization $y = \exp{(z)}$, we perform a translation

$$\mathcal{R}_{h\ 12}(z) \to \mathcal{R}_{h\ 12}(z - 2h\,X_{(3)}) \tag{42}$$

to express (35) in a symmetric form

$$\begin{aligned}&\mathcal{R}_{h\ 12}(z-2hX_{(3)})\,\mathcal{R}_{h\ 13}(z+2hX_{(2)})\,\mathcal{R}_{h\ 23}(z-2hX_{(1)})\\ &\qquad = \mathcal{R}_{h\ 23}(z+2hX_{(1)})\,\mathcal{R}_{h\ 13}(z-2hX_{(2)})\,\mathcal{R}_{h\ 12}(z+2hX_{(3)}).\end{aligned} \tag{43}$$

This is equivalent to the Jordanian GNF equation (35) for the class of representations $\varrho_{j_1;\,j_2}$ satisfying the property

$$\varrho_{j_1;\,j_2}\Big(\Big[(X_{(k)}+X_{(l)})\,\partial_z, \mathcal{R}_{h\ kl}(z)\Big]\Big) = 0. \tag{44}$$

Adopting the procedure in [8] we here use the following construction of the Lax operator for the $U_{h;y}(sl(2))$ algebra

$$L_{13}(z) = \exp\Big[-2h\,(2X_{(1)}+X_{(3)})\,\partial_z\Big]\;\mathcal{R}_{h\ 13}(z)\,\exp\Big[2h\,X_{(3)}\,\partial_z\Big], \tag{45}$$

where the subscript 3 denotes the quantum space. For the representations satisfying (44) the relation (43) may be expressed in a Lax martix form

$$R_{h\ 12}^{j_1;\,j_2}(z-2h\,X_{(3)})\,L_{13}(z)\,L_{23}(z) = L_{23}(z)\,L_{13}(z)\,R_{h\ 12}^{j_1;\,j_2}(z+2h\,X_{(3)}). \tag{46}$$

As illustrations we note that the representations $R_h^{\frac{1}{2}\,;\,1}(z)$, $R_h^{1\,;\,\frac{1}{2}}(z)$ and $R_h^{1\,;\,1}(z)$ obtained from (38) and (39) satisfy the requirement (44).

To summarize, here we have constructed the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra by explicitly obtaining the relevant twist operator via a contraction method. In the contraction method used here we start with the standard Drinfeld-Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra and use a suitable similarity transformation followed by a $q \to 1$ limiting process. An important point here is that the reparametrization as obtained in (22) is essential for obtaining a *nonsingular* twist operator for the $U_{h;y}(sl(2))$ algebra in the $q \to 1$ limit. Our contraction method has an advantage in that it furnishes the dynamical quantities for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra from the corresponding quantities of the standard

Drinfeld-Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra. The present twist operator associated with the $U_{h;y}(sl(2))$ algebra satisfies a shifted cocycle condition. The universal $\mathcal{R}_h(y)$ matrix satisfies the GNF equation associated with the $U_{h;y}(sl(2))$ algebra. For a special class of representations, the GNF equation may be recast as a compatibility condition of the L operators. As an extension of the present work, a similar formalism may be developed to describe a quasi-Hopf quantization of the coloured Jordanian deformed $gl(2)$ algebra considered in [15], [16], [4]. A similar construction of the twist operators associated with the quasi-Hopf deformation of the Jordanian $sl_h(N)$ algebra may also be attempted following the discussion in [2].

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References

[1] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, Mod. Phys. Lett.**A11** (1996) 2883.

[2] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, Mod. Phys. Lett.**A13** (1998) 779.

[3] B. Abdesselam, A.Chakrabarti, R.Chakrabarti and J. Segar, Mod. Phys. Lett.**A14** (1999) 765.

[4] R. Chakrabarti and C. Quesne, Int. J. Mod. Phys. **A14** (1999) 2511.

[5] P. P. Kulish, V. D. Lyakhovsky and A. I. Mudrov, *Extended Jordanian twists for Lie algebras*, **math.QA**/9806014.

[6] J. L. Gervais and A. Neveu, Nucl. Phys. **238** (1984) 125.

[7] G. Felder, *Elliptic Quantum Groups*, Proc. ICMP, Paris (1994).

[8] O. Babelon, D. Bernard and E. Billey, Phys. Lett. **B375** (1996) 89.

[9] C. Fronsdal, Lett. Math. Phys. **40** (1997) 117.

[10] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, **q-alg**/9712029.

[11] D. Arnaudon, E. Buffenoir, E. Ragoucy and Ph. Roche, *Universal solutions of Quantum Dynamical Yang-Baxter equations*, **q-alg**/9712037.

[12] C. Kassel, *Quantum groups*, (1995) Springer Verlag.

[13] Ch. Ohn, Lett. Math. Phys. **25** (1992) 85.

[14] A. Ballesteros and F. J. Herranz, J. Phys. A: Math. Gen. **29** (1996) L311.

[15] C. Quesne, J. Math. Phys. **38** (1997) 6018.

[16] P. Parashar, Lett. Math. Phys. **45** (1998) 105.